What Do Students Struggle with When First Introduced to Algebra Symbols?

Up until fairly recently, algebra was considered an exclusively letter-symbolic domain. Much of the research that was conducted during the period preceding the 1990s focused especially on the transitions required by students as they moved from arithmetic to algebra at about age thirteen or fourteen (see Kieran 1992). This large body of research, which continues to grow (see Kieran 2007), has studied the learning of the concepts that underpin students' success in algebra. These concepts include unknowns and variables, expressions and equations, and the expansion of the meaning given to the equal and minus signs.

Some of the earliest research focused on students' ability to discriminate among the different ways that letters are used in algebra. It was found that only a small percentage of students were able to consider algebraic letters as generalized numbers or as variables, with the majority interpreting letters as specific unknowns. For example, Küchemann (1981) noted in his research that although most students could not handle questions such as "Which is larger, $2n$ or $n + 2$?" they had no difficulty with questions of the type "If $a + b = 43$, then $a + b + 2 = ?$" However, changing curricular emphases and using technological tools have expanded students' views of algebraic letters. For example, spreadsheet activity has been found to encourage simultaneous multi-valued and single-valued interpretations of the letter (Ainley 1996). The shift in the content of algebra from an equation-centered to a function-centered content has broadened students' views of algebraic letters, but it has also introduced additional difficulties, according to some research. For example, a function in two variables (e.g., $y = 3x + 10$) becomes an equation in a single unknown as soon as a value for y is fixed (e.g., $100 = 3x + 10$), which has been found to create some confusion on the part of students as functions and equations tend to become fused (Chazan and Yerushalmy 2003). It is not that such dilemmas, which result from combining functional approaches with more standard treatments of school algebra, cannot be resolved; however, *students need to be provided with opportunities to coordinate their knowledge of unknowns in equations with their understandings of variables in functions.*

The research dealing with the learning of algebraic expressions and equations has investigated students' thinking in three areas: (1) word problems leading to an algebraic equation, (2) geometric and numerical patterns leading to an algebraic expression, and (3) numerical relationships leading to an algebraic expression.

The large body of research treating the representation of word problems by equations continues to show that *students prefer to solve word problems by arithmetic reasoning* rather than first representing the problem by an algebraic equation and then applying algebraic transformations to that equation. For example, Stacey and MacGregor (1999) found that many students relied on arithmetic approaches even in problems where they were specifically encouraged to use algebraic methods, as in the following:

> Some money is shared between Mark and Jan so that Mark gets \$5 more than Jan gets. Jan gets \$$x$. Use algebra to write Mark's amount. The money to be shared is \$47. Use algebra to work out how much Jan and Mark would get.

Students often used arithmetic in this problem: "Take out \$5 first, give it to Mark, then share the rest equally" (p. 156). Setting up the equation requires an analytic mode of thinking that is exactly opposite to that used when solving a problem arithmetically (Kieran and Chalouh 1993). In fact, when permitted to choose their own solving methods, students find word problems presented in verbal form easier to solve than comparable questions presented in other forms, such as equations, or "word-equations" (Nathan and Koedinger 2000). Although earlier research in the 1970s and 1980s had characterized students' activity with algebra word problems by means of the development of generalized schemata for different problem types (e.g., mixture problems, age problems, distance problems), current curricula reflect a movement away from such traditional word problems. The emphases of present-day curricula highlight the motivating role of more realistic problem situations within algebra instruction; however, *students' persistent difficulties with the framing of equations to represent word-problem situations lead to questions regarding the feasibility of such approaches for developing algebraic competence.*

The research on the use of patterning activities to develop meaning for algebraic expressions suggests that hard work is needed by students in order for them to express the observed numerical and geometric patterns in a letter-symbolic form (Healy and Hoyles 1999). The ways in which students attend to certain perceptual aspects of a pattern can make it difficult for them to express generality, either verbally or symbolically. For instance, when MacGregor and Stacey (1993) presented function tables to seventh-grade students to see what patterns and relationships they would perceive (see, for example, the table in fig. 1),

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	11
8	..
..	..

Fig. 1

the researchers found that some students noticed relationships that were not helpful, such as adding the pairs of *x*- and *y*-values in each row to yield a recurrence pattern in the totals: "When you're adding both numbers they are always increasing, like 1 and 5 is 6, 2 and 6 is 8, 3 and 7 is 10, … each numbers go up by 2."

Furthermore, even with students in eighth and ninth grades, Radford (2006) has noted that their use of symbols within patterning activities does not always amount to "doing algebra." Radford distinguishes algebraic from arithmetic generalization: "Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some elements of a sequence *S*, being aware that this commonality applies to all the terms of *S* and being able to use it to provide a direct *expression* of whatever term of *S*" (p. 4). Arithmetic generalizations are those that do not involve a rule that provides one with an expression of "whatever term" of the sequence. For example, when Radford presented a toothpick pattern (see fig. 2) to small groups of eighth-grade students, who were then asked how many toothpicks would be in figure 25, some reasoned as follows: "The next figure has two more than … look … […] [Figure] 6 is 13, 13 plus 2. You have to continue there […]" (p. 7).

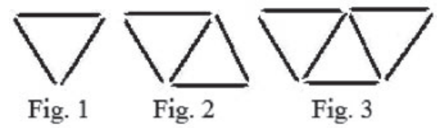

Fig. 2

Radford has pointed out that the students did notice that the terms of the sequence increase by 2 and that this common increment applies to the terms that followed. In other words, they did generalize something, but they remained in the realm of arithmetic. What they generalized was a local commonality observed on some figures, without being able to use this information to supply an expression for the 25th, or whatever, term of the sequence. *Moving from arithmetic to algebraic generalizations is a process that has been found to take time.*

Whether algebraic expressions are generated from patterns or from other types of numerical activity, it is not long before the expressions are combined to form equations. The meaning that most students have given to the equal sign in arithmetic must then be extended (Kieran 1981). Much of elementary school arithmetic is answer oriented. Students who interpret the equal sign as a signal to compute the left side and then to write the result of this computation immediately after the equal sign might be able to correctly interpret algebraic equations such as $2x + 3 = 7$ but not equations such as $2x + 3 = x + 4$. *Experience with the construction of multioperation arithmetic equalities* (e.g., $7 \times 2 + 3 - 2 = 5 \times 2 - 1 + 6$) and with justifying such "arithmetic identities" with the same total value on both sides *has been found to help in extending students' meaning of the equal sign from a do-something signal to a relational symbol* (Herscovics and Kieran 1980). Subsequent experience in covering up any one of the numbers (on one or both sides) of these arithmetic identities can then be used to make the transition to algebraic equations with letters on both sides of the equal sign. *Another important adjustment to be made by students if algebraic reasoning is to develop involves the meaning given to the minus sign.* It has to be extended from a sign denoting subtraction to include "negative something" as in –3, and "the opposite of" as in –x. Research has shown that students' difficulties in understanding the many roles that the minus sign plays in algebra can persist for long periods of time (Vlassis 2001).

During their introduction to symbolic activity, students attempt to give meaning to unknowns and variables, expressions, and equations and extend their understandings of the equal and minus signs. It is crucial for students' success in algebra that they make sense of these concepts and be able to use these symbols to express generality. The research discussed in this brief points to how students struggle to understand these important ideas and what activities might be carried out in the classroom to strengthen such understanding.

By Carolyn Kieran
Judith Reed, Series Editor

REFERENCES

Ainley, Janet. "Purposeful Contexts for Formal Notation in a Spreadsheet Environment." *Journal of Mathematical Behavior* 15 (1996): 405–22.

Chazan, Daniel, and Michal Yerushalmy. "On Appreciating the Cognitive Complexity of School Algebra: Research on Algebra Learning and Directions of Curricular Change." In *A Research Companion to "Principles and Standards for School Mathematics,"* edited by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter, pp. 123–35. Reston, Va.: National Council of Teachers of Mathematics, 2003.

Healy, Lulu, and Celia Hoyles. "Visual and Symbolic Reasoning in Mathematics: Making Connections with Computers?" *Mathematical Thinking and Learning* 1 (1999): 59–84.

Herscovics, Nicolas, and Carolyn Kieran. "Constructing Meaning for the Concept of Equation." *Mathematics Teacher* 73 (November 1980): 572–80.

Kieran, Carolyn. "Concepts Associated with the Equality Symbol." *Educational Studies in Mathematics* 12 (1981): 317–26.

———."The Learning and Teaching of School Algebra." In *Handbook of Research on Mathematics Teaching and Learning,* edited by Douglas A. Grouws, pp. 390–419. Reston, Va.: National Council of Teachers of Mathematics, 1992.

———."Learning and Teaching Algebra at the Middle School through College Levels: Building Meaning for Symbols and Their Manipulation." In *Second Handbook of Research on Mathematics Teaching and Learning,* edited by Frank K. Lester, Jr., pp. 707–62. Charlotte, N.C.: Information Age Publishing; Reston, Va. National Council of Teachers of Mathematics, 2007.

Kieran, Carolyn, and Louise Chalouh. "Prealgebra: The Transition from Arithmetic to Algebra." In *Research Ideas for the Classroom: Middle Grades Mathematics,* edited by Douglas T. Owens, pp. 179–98. New York: Macmillan Publishing Co., 1993.

Küchemann, Dietmar. "Algebra." In *Children's Understanding of Mathematics: 11–16,* edited by K. M. Hart, pp. 102–19. London: John Murray, 1981.

MacGregor, Mollie, and Kaye Stacey. "Seeing a Pattern and Writing a Rule." In *Proceedings of the 17th Conference of the International Group for the Psychology of Mathematics Education* (PME), vol. 1, edited by Ichiei Hirabayashi, Nobuhlko Nohda, Keiichi Shigematsu, and Fou-Lai Lin, pp. 181–88. Tsukuba, Japan: PME, 1993.

Nathan, Mitchell J., and Kenneth R. Koedinger. "Teachers' and Researchers' Beliefs about the Development of Algebraic Reasoning." *Journal for Research in Mathematics Education* 31 (March 2000): 168–90.

Radford, Luis. "Algebraic Thinking and the Generalization of Patterns: A Semiotic Perspective." In *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* edited by Silvia Alatorre. [CD-ROM]. Merida, Mexico: PME-NA, 2006.

Stacey, Kaye, and Mollie MacGregor, "Learning the Algebraic Method of Solving Problems." *Journal of Mathematical Behavior* 18 (1999): 149–67.

Vlassis, Joelle. "Solving Equations with Negatives or Crossing the Formalizing Gap." In *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (PME), vol. 4, edited by Marja van den Heuvel-Panhuizen, pp. 375–82. Utrecht, Netherlands: PME, 2001.