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Developing Students' Representational Fluency Using Virtual and Physical Algebra Balances

JENNIFER SUH AND PATRICIA S. MOYER
George Mason University
USA
jsuh4@gmu.edu
pmoyer@gmu.edu

Both virtual and physical manipulatives are reported as effective learning tools when used with different groups of students in a variety of contexts to learn mathematical content. The use of multiple representations and the flexibility to translate among those representational forms facilitates students' learning and has the potential to deepen their understanding.

This classroom project involved two groups of third-grade students in a week-long unit focusing on algebraic relationships. The purpose of the unit was to engage students with different algebraic models and encourage students to use informal strategies to represent their relational thinking. The paper highlights examples of these student representations as evidence of the children's developing algebraic thinking.

Result from the pre and posttest measures showed that students in the physical and virtual manipulative environments gained significantly in achievement and showed flexibility in translating and representing their understanding in multiple representations: manipulative model, pictorial, numeric, and word problems. The researchers recorded field notes, interviewed students, and videotaped class sessions in order to identify unique features of the learning environments. The virtual environment had unique features that promoted student thinking such as: (a) explicit linking of visual and symbolic modes; (b) guided step-by-step support in algorithmic processes; and (c) immediate feedback and self-checking system. In the physical environment, some unique features were: (a) tactile features; (b) opportunities for invented strategies; and (c) mental mathematics. These results show that although the different manipulative models had different features, both the physical and virtual environments were effective in supporting students' learning and encouraging relational thinking and algebraic reasoning.

Students in the elementary grades often use language to describe their thinking that includes elements of algebraic reasoning, even though more formalized study of algebra does not occur until much later in the school curriculum. Through meaningful investigations, these conversations and ideas can be expressed using multiple representations, such as physical and virtual manipulatives, drawings, and symbolic expressions with variables. These representations and informal ideas can form the basis for students' relational thinking and algebraic reasoning.

This article describes an algebra unit that contained open-ended activities providing access to algebra concepts for third-grade students. During the project, students used mathematical models, including physical and virtual manipulatives, and pictorial and written expressions, to represent quantitative relationships. The unique characteristics of different representational forms allowed students to express their algebraic thinking in a variety of ways. The article highlights examples of these student representations as evidence of the children's developing algebraic thinking.

IMPORTANCE OF MULTIPLE REPRESENTATIONS IN MATHEMATICS

The *Principles and Standards for School Mathematics* (NCTM, 2000) emphasize the role of representation in mathematics stating that students should "…create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representation to solve problems; and use representations to model and interpret physical social and mathematical phenomena" (p. 67). Theorists on representation concur with NCTM's goals. For example, Greenes and Findell (1999) stated that students develop mathematical reasoning in algebra when they are able to interpret algebraic equations using pictorial, graphic, and symbolic representations. They recommend experiences, such as representing algebraic expressions using balance scales, to promote students' relational thinking.

Other theorists also suggest the importance of students learning using multiple modes. Gardner (1993) in *Multiple Intelligences: The Theory in Practice* recommended that curriculum material be presented in multiple modes to capitalize on personal learning styles. For example, in a geometry class, teachers should "draw upon spatial, logical, linguistic, and numerical competences" (p. 73). Lesh, Landau, and Hamilton (1983) emphasized the importance of translation among mathematical representations (Figure 1). They identified five distinct types of representation systems: (a) real life experience, (b) manipulative models, (c) pictures or diagrams, (d) spoken symbols, and (e) written symbols (p. 265).

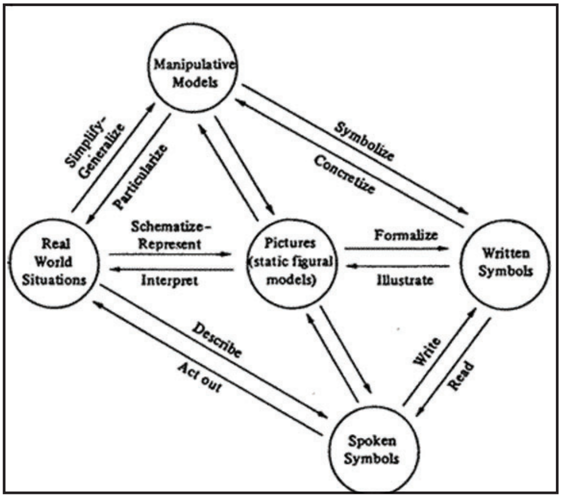


Figure 1. Five distinct types of representation system (Lesh, Landau, & Hamilton, 1983)

Cramer (2003) described representational fluency in the Lesh, Landau, and Hamilton (1983) model as follows: "The model suggests that the development of deep understanding of mathematical ideas requires experience in different modes, and experience making connections between and within these modes of representation. A translation requires a reinterpretation of an idea from one mode of representation to another" (p. 1). This type of translation can support students' relational thinking and algebraic reasoning. Although distinct types of representational systems are important, the ability to translate among different modes of representation indicates deeper conceptual understanding within the system. Asking students to restate problems in their own words, draw diagrams to illustrate problems, or act out problems are ways students can demonstrate flexibility in translating among representations.

Research outside the field of mathematics education also supports these ideas. *Dual Coding,* proposed by researchers in the field of educational psychology and based on Cognitive Information Processing Theory, is the assumption that information for memory is processed and stored by two interconnected systems and sets of codes-verbal codes and visual codes (Clark & Paivio, 1991). Rieber (1994) reported that it is easier to recall information from the visual processing codes than the verbal codes because visual information is accessed using synchronous processing, rather than sequential processing. Rieber noted, "adding pictures (external or internal) to prose facilitates learning, assuming that the pictures are congruent to the learning task;" and, "children do not automatically or spontaneously form mental images when reading" (p.141). Applying dual coding theory to mathematics education, information that makes this representational connection between verbal and visual (pictures) forms is easier to retain and retrieve because two mental representations are available rather than one.

THE USE OF PHYSICAL AND VIRTUAL REPRESENTATIONS IN CLASSROOMS

The use of manipulatives as a physical representation during mathematics instruction has been discussed for decades in the mathematics education literature. Balka (1993) described the benefit of using manipulatives by stating, "The use of manipulatives allows students to make the important linkages between conceptual and procedural knowledge, to recognize relationships among different areas of mathematics, to see mathematics as an integrated whole, to explore problems using physical models, and to relate procedures in an equivalent representation" (p. 22). However, Kaput (1989) expressed the caution that students do not automatically make the connection between their actions with the manipulatives and their actions with symbols. One possible explanation for this disconnect is that the cognitive load imposed during the activity with the manipulatives is too great for students. In essence, students are unable to track all of their actions with the manipulatives and fail to see the connection between these actions and the actions that they take on symbols.

A small but growing body of classroom research has begun to emerge on uses of virtual manipulatives as a representation for mathematics instruction. Virtual manipulatives have been defined as “computer based renditions of common mathematics manipulatives and tools” (Dorward, 2002, p.329) and “an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (Moyer, Bolyard, & Spikell, 2002, p. 373). The NCTM (2000) stated that “work with virtual manipulatives can allow young children to extend physical experience and to develop an initial understanding of sophisticated ideas like the use of algorithms” (p. 26-27). An appealing characteristic of these new technological tools is on their interactivity and capability to present multiple representations at the same time on the computer screen. For example, some virtual manipulative applets present users with dynamic visual objects, symbolic or numeric expressions, and written explanations simultaneously on the screen. This enables the user to have parallel views of both visual and written expressions of the same mathematical concept or process.

Although research on virtual manipulatives is limited, there are promising classroom studies and dissertations that demonstrate the unique features of these tools for teaching mathematics. Terry’s (1996) study of 102 students in grades two through five using base 10 blocks and attribute blocks found that when students used a combination of both physical and virtual manipulatives, they showed significant gains between the pretest and posttest when compared to students using only physical manipulatives or virtual manipulatives. Takahashi’s (2002) dissertation, using a physical geoboard and a virtual geoboard with middle school students also indicated that students benefited from instruction when both types of tools were used.

Reimer and Moyer (2005) reported on action research in a third grade classroom using virtual manipulatives to learn about fractions. Reimer taught 19 third grade students for two weeks using several interactive virtual fraction manipulatives. Task sheets were provided to students on each day that they worked with the virtual manipulatives in the computer lab. Data were collected from pretests and posttests of students’ conceptual knowledge and procedural computation, student interviews, and attitude surveys. The results indicated a statistically significant improvement in students’ conceptual knowledge and a significant positive relationship between students’ scores on the posttests of conceptual and procedural knowledge. Student attitude surveys indicated that the virtual manipulatives helped them learn by providing immediate and specific feedback, being faster to use than paper and pencil methods, and enhancing students enjoyment while learning fractions.

Suh’s (2005) dissertation showed statistically different achievement results in a unit on fraction addition where one group used virtual fraction applets and the other group used physical fraction manipulatives. This research highlighted how different representations, such as physical and virtual manipulatives, can have unique features that promote different kinds of learning mathematical concepts. Moyer, Niezgoda, and Stanley's (2005) research on kindergarten students' construction of patterns also promoted this idea. In this study, the use of different physical, virtual, and pictorial representations resulted in students creating a variety of repeating and growing patterns, patterns with different levels of complexity, and evidence of creative behaviors. These studies show that it is important to look beyond pre and posttest information to examine the characteristics of different learning environments and how those characteristics influence different types of learning experiences.

Features of both virtual and physical manipulatives have been found to be beneficial when used with different groups of students in different contexts for different mathematical content. This study adds to this research base by examining how exposure to multiple representations influences students' responses on pictorial, symbolic, and written test items when students are exploring algebraic concepts.

THIRD GRADERS EXPLORE ALGEBRA USING MULTIPLE REPRESENTATIONS

This classroom project involved two groups of third-grade students in a week-long unit focusing on algebraic relationships. The purpose of the unit was to engage students with different algebraic models and encourage students to use informal strategies to represent their relational thinking. During the unit, 36 third graders worked with virtual and physical balance scales during their regularly scheduled mathematics class sessions.

Group One worked with the Virtual Balance Scale applet on the National Library of Virtual Manipulatives (http://matti.usu.edu/nlvm/nav/) to solve simple linear equations. The unit block, representing 1 and a blue x-box, representing the unknown x, are placed on the pans of a balance scale. Once the beam balances to represent the given linear equation, students can choose to perform any arithmetic operation, as long as they perform the same operation on both sides of the equation, thus keeping the pans balanced. If the equation is not balanced, the beam will slant to one side. The goal of the applet is to get a single x-box on one side, with the amount needed for balance on the other side, thus giving the value of x (Figure 2).

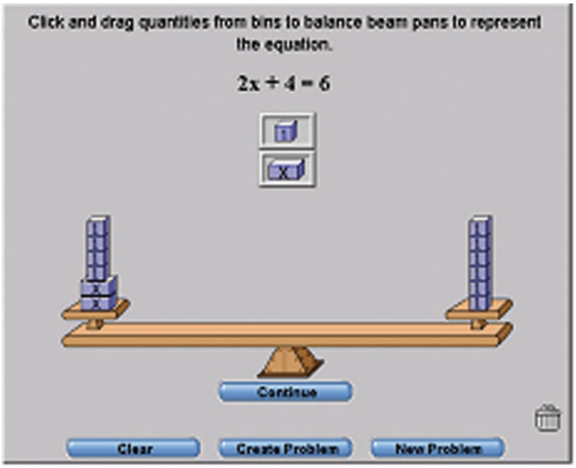


Figure 2. Virtual manipulative algebra balance scale

Group Two worked with a physical manipulative called Hands-On Equations® (Borenson, 1997). These materials are a visual and kinesthetic teaching system for introducing algebraic concepts to students in grades three to eight. The teacher's edition comes with a stationary plastic balance scale, number cubes, and pawn pieces. The student edition comes with a balance scale mat, number cubes and pawn pieces to represent algebraic equations (Figure 3). The pawn pieces represent the unknown *x* value and the number cubes represent numbers in the equation. When a student removes a pawn or a number from one side, the same number of pawns or number cubes must be removed from the other side of the balance scale to keep the equation balanced.

Figure 3. Hands-On Equations®

During the algebra unit, students in both environments were given opportunities to work with multiple representations to build representational fluency. For example, students worked to translate word problems to pictures, pictures to manipulative models, manipulative models to algebraic expressions, and algebraic expressions to written and verbal explanations. An example of a word problem from the unit follows:

> Mrs. Lee wanted to see how much her baby weighed but the baby could not sit or stand on the scale by himself so she decided to get on the scale with him. When she stepped up, the scale read 150 pounds. Mrs. Lee knew she weighed 130 pounds. How much did her baby weigh?

As a class, the teacher and students worked together to show how this problem could be written as an equation (150 = 130 + B, where B stands for the baby's weight). Students were shown simple models of algebraic relationships using arithmetic sentences (i.e., 2 + 3 = 5) on the balance scale. This was done to highlight that the equal sign is relational, rather than operational, which is a common misconception for elementary students. When the teacher introduced the idea of x as the unknown, she used a box with an x written on it and placed it over the number 3 to represent the missing addend. She wrote $2 + x = 5$ and asked students to determine the value of the unknown.

During the unit, both groups had opportunities to translate algebraic expressions into manipulative models. Students in the physical manipulative group completed task sheets with several algebraic equations that they modeled using the Hands on Equation balance scale mat. Students using the virtual balance scale set up the algebraic expressions shown on the computer screen and used the blocks to solve for x. Both the Hands-On Equations® manipulatives and the virtual balance applet helped students represent the written expressions of quantitative relationships using manipulative models. However the link between the symbolic and manipulative representations was more closely tied together in the virtual manipulative environment, because the symbolic expression was on the screen during the process. Students in both groups kept a record of their mathematical procedures using drawings and written expressions (Figures 3 and 4). Although, both groups were actively engaged in recording their work with the physical and virtual manipulatives, the nature of the task sheets differed in some ways. For example, as shown in Figure 4, while working with the Hands-On Equations® , students translated the pictorial representations into algebraic expressions and wrote the expressions above the pictures. They were asked to use arithmetic operations to check their answers.

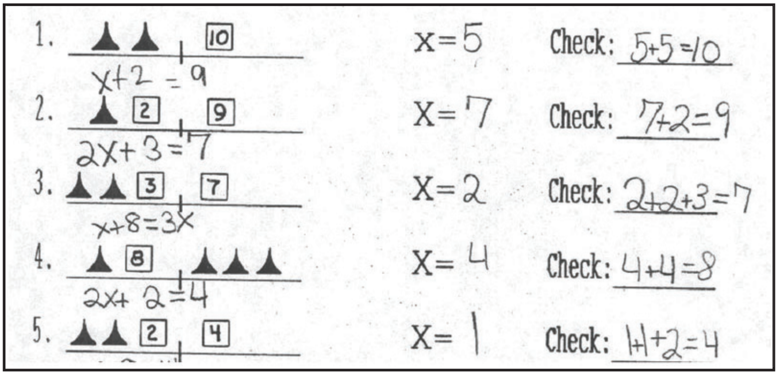


Figure 4. A portion of a student's activity sheet for Hands-On Equations®

In the virtual manipulatives group, students worked on the problems online and recorded the arithmetic operations on the task sheet while working with the balance scale applet (Figure 5). After several problems, students were asked to print out three screen shots of the work from the virtual applet. Students in both groups were asked to verbalize the steps they took to solve the problems. At the end of each lesson the teacher closed with a whole-class discussion that focused on questions such as: (a) What were some strategies you used to find the value of *x*? (b) How would you describe the rules for finding the value of *x* to someone who doesn't know algebra?

Students in both third-grade classes completed a pretest and posttest during the unit that contained different types of test items: pictorial, numeric, and word problems. These items were used to compare student progress and use of different representational forms. The researchers recorded field notes, interviewed students, and videotaped class sessions in order to identify unique features of the learning environments. The purpose for collecting data using a variety of sources was to document student learning, as well as to examine students' uses of various representations and solution strategies while using the physical and virtual balance scales.

Unique Features of the Representations that Promoted Student Learning

During class sessions, it was evident that students were developing their ability to represent ideas in their drawings and written work. From the observational field notes and student interviews, the researchers discovered several features that were unique to the physical and virtual environments

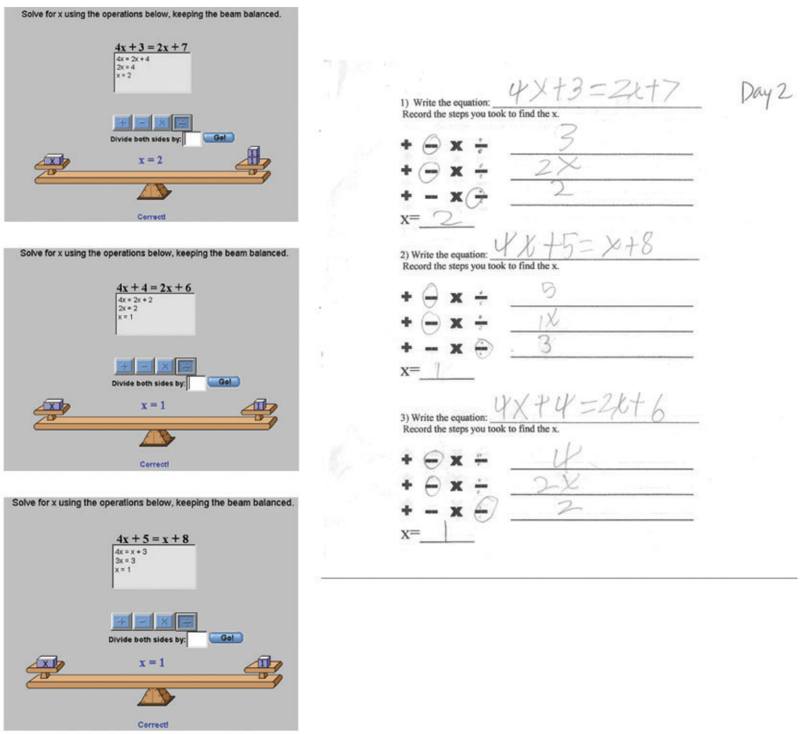
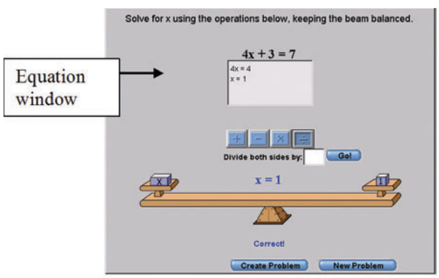


Figure 5. Screenshot of student's problem and task sheet for recording work

that promoted learning. In the physical environment, some unique features of the Hands-On Equations manipulatives that were distinct from the virtual manipulatives were: (a) tactile features; (b) more opportunities for invented strategies; and (c) more mental mathematics. In the physical environment, students could pick up and move the algebra pieces easily without a mouse. Their movements were more efficient with the algebra pieces. Because the physical environment was more open ended and did not provide students with a guided process for solving the equations, this encouraged some students to invent their own strategies for solving equations, rather than following a traditional algorithm. Some students relied on multiplication instead of using division to find the value of x when there was a missing factor. For example, in the equation $3x = 6$, students said, "What times 3 gives me 6?" instead of thinking "6 divided by 3 equals 2." The open-ended feature of the Hands-On Equations also prompted students to use more mental mathematics in their calculations and allowed them to process numerical relationships mentally before writing down their ideas. This was evident when students were working and they would not record the calculations on paper, instead choosing to move the algebra pieces physically and talk aloud until they arrived at a solution. Other students used a guess and check strategy by substituting a number for *x* to see if it was correct.

The virtual environment also had unique features that promoted student thinking such as: (a) explicit linking of visual and symbolic modes; (b) guided step-by-step support in algorithmic processes; and (c) immediate feedback and self-checking system. One of the features of the virtual balance scale was that it explicitly linked a dynamic picture of the balance scale with the symbolic representation of the algebraic equations that were presented on the scale. When students typed in a symbolic command such as "subtract 3*x* from both sides," the dynamic feature of the applet removed three of the *x* boxes from both sides of the balance scale and simultaneously displayed a new equation on the screen. The equation window tracked moves made by the student, thereby scaffolding the process of solving for *x*, and explicitly providing the connection between the equations and the actions of the balance scale. During class sessions, when the teacher asked students to explain their solution processes, students were observed using the equation window, which is where these processes had been recorded by the virtual applet (Figure 6).



1) After setting up the problem, students clicked on the minus sign and subtracted 3 boxes from both sides.

2) Once the 3 boxes disappeared from the screen, students chose the division sign and divided by 4 from each side.

3) Each step taken by the student (algorithm) was displayed on the equation window.

Figure 6. Guided step by step support with formal algorithm

Another feature of the virtual balance applet was a built in constraint-support system that emphasized the guided, step-by-step process for solving the equations. The balance scale placed an emphasis on subtraction and division as solution routes for balancing the equations. For example, guidance would be given such as, "You can't subtract 4*x* from both sides unless there are at least 4*x*s on each side." Because of these features, students were required to choose an operation and perform the operation while the applet displayed each equation during the solution process. Teaching cues were provided to ensure that students performed the procedures accurately. Students received immediate feedback while they were solving the problems and were able to use a self-checking feature to determine the accuracy of their solutions. For example, if students made an error, the computer would prompt, “The two sides don’t match the equation.” This self-checking system kept students from practicing erroneous solution routes and allowed them to check their own answers. Students liked the way the balance scale tilted and balanced based on the equations. They commented, “I like the way the balance scale shows me I have set up the right number sentence by balancing itself. If I don’t do it right, one side slants down.” The teacher’s observational notes also highlighted this feature: “One advantage that I saw with this tool was that the balance scale tilted as blocks were removed. This feature showed students the inequality and equality of an equation by the tilt of the balance scale.”

Analysis of Students’ Algebraic Reasoning

The project team collected and analyzed data from the pre and posttests. A paired samples *t*-test revealed that both groups showed significant gains during the unit. These values are shown in Table 1.

Table 1

Pretest and Posttest Results

Manipulative Groups	*M* Pretest	*M* Posttest	*M* Differences	*SD*	*T* value	Sig.
Virtual (Group 1)	30.00	83.33	53.33	17.32	13.06	.000 ***
Physical (Group 2)	21.66	80.55	58.88	21.32	11.71	.000 ***

*** =a significant at .000 level

As these results indicate, students in the physical and virtual manipulative environments showed significant gains in achievement between the pre and posttest measures. Although the environments had different features, both representations were effective in supporting students’ learning in different ways.

In addition to these overall results, researchers were interested in students’ performance on three different item types (8 pictorial, 8 symbolic, and 2 word problems) of the posttest. The team used a scoring rubric to categorize the level of students’ understanding on these items. Table 2 shows means for each section of the posttest.

Table 2
Posttest Means for Different Test Item Types

Manipulative Groups	Mean on Pictorial	Means on Symbolic	Means on Word Problems
Virtual (Group 1)	94.44 (SD=8.80)	75.00 (SD=26.42)	83.33 (SD=24.25)
Physical (Group 2)	90.27 (SD=15.78).	77.50 (SD=20.10).	80.00 (SD=22.00)

ALGEBRA POSTTEST PICTORIAL SECTION

Students' written work on the test items revealed that there was a number of interesting solution strategies. Students' scores were highest on items with pictorial representations. In this section, students used a given picture and made marks on the picture as a way to cross off quantities from both sides of the balance scale in the picture. (Figure 7 shows how a student used pencil marks to cross off quantities.)

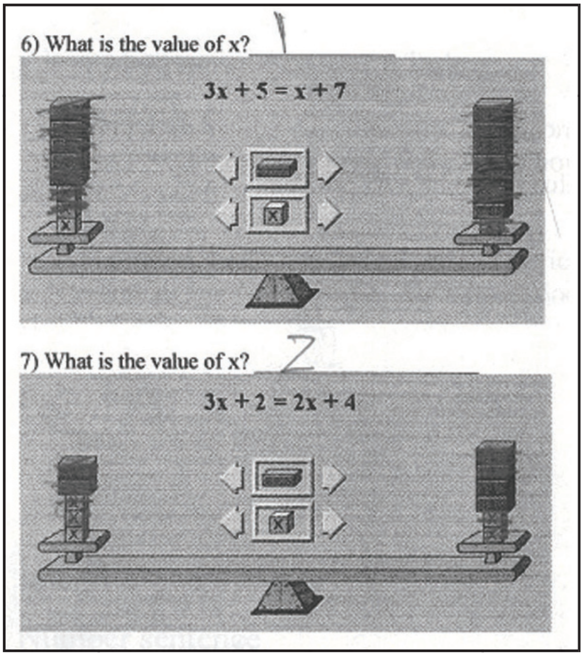


Figure 7. Example of student using comparing quantities strategy on pictorial items

Some of the students in the virtual manipulative group, wrote a simplified algebraic equation above the problem and also crossed off values on the picture. (See Figure 8 for an example of this solution.) This was a process that was contained in the equation window on the virtual balance applet.

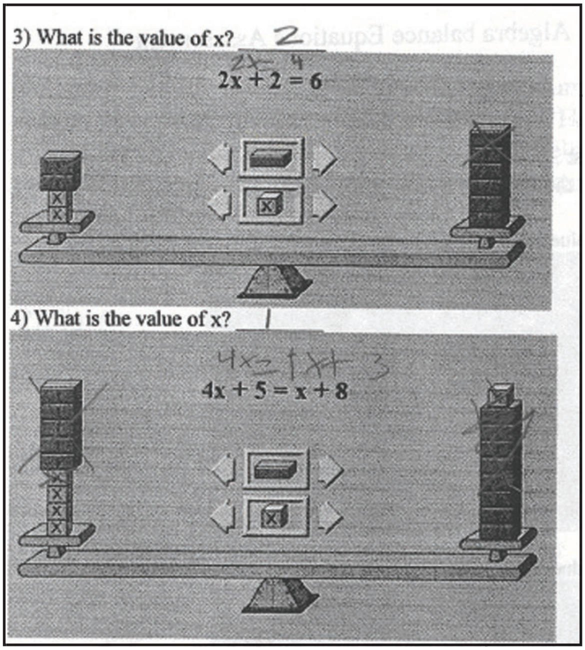


Figure 8. Example of student using symbolic algorithm

Algebra Posttest Symbolic Section

Students' primary solution strategies for the items presented symbolically were drawings accompanied by calculations using subtraction (Figure 9). The directions on this section of the posttest stated: "Find what the value of *x* is in each problem." Although, the posttest did not explicitly direct students to provide a drawing for the response, most students drew pictures that resembled the balance scale model and used these drawings to help them solve the problems. As Figure 9 shows, students' drawings were not an exact match for the balance scale model. However, all of their drawings show an understanding of the concept of equivalence for balancing the equation and an understanding of the difference between 3*x* and the number 3. This can be seen in the student's drawing where three letter *x* s are used to represent 3*x*.

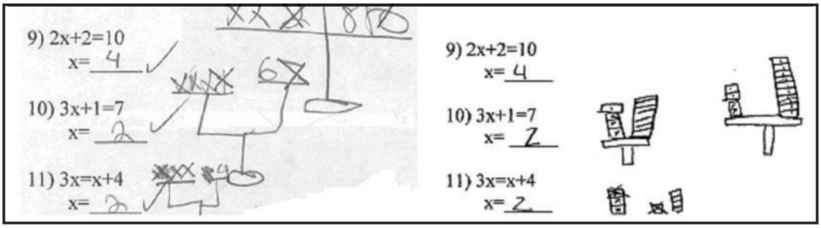


Figure 9. Students' pictorial representations for numeric equations

Table 3 shows a summary of the types of solutions used by students in both groups. Across both groups of students, 28 of 36 third graders relied on pictorial representations to help them find the value of *x*. A small number of students relied on the use of the formal algorithm to solve the symbolic questions or they did not show any strategy on their papers. The use of both representation forms appeared to encourage students to illustrate and translate the symbolic expressions into pictorial representations when solving the problems. Figure 9 shows drawings from both groups.

Table 3
Analysis of Solution Strategies from the Symbolic Items on the Algebra Posttest

Solution strategies	Group One: Virtual		Group Two: Physical	
	N Students	Percent	N Students	Percent
Used primarily pictorial	14	78 %	14	78%
Used primarily algorithms	3	16%	2	11%
No strategy shown	1	6%	2	11%

ALGEBRA WORD PROBLEM ITEMS

The researchers analyzed students' responses to the word problems by determining the number of students who translated the word problems into (a) pictorial representations only, (b) pictorial and symbolic representations, and (c) symbolic representations only (Table 4). Most students (28 of 36) translated the word problems into pictorial representations and number sentences as requested on this section of the posttest.

Table 4
Translating Word Problem into Other Representational Forms

Representational forms	Group One: Virtual Algebra Balance		Group Two: Physical Hands-On Equations®	
	N Students	Percent	N Students	Percent
Pictorial Representations Only	1	5.5%	4	22%
Pictorial and Symbolic Representations	16	89%	12	67%
Symbolic Representations	1	5.5 %	2	11%

Students' drawings revealed that they were able to translate the balance scale representation of "two equivalent amounts on each side of the equation" to pictures that contained one quantity on the left, one on the right, and an equal sign between the two. An example of this concept is shown in Figure 11, where the student has drawn pizzas and drinks on two imaginary pans of an imaginary balance scale with the equal sign between the two quantities. Another example of students' developing algebraic thinking appears in the response in Figure 11. Here, the student wrote the letter *p* to stand for pizzas and *o* or circles to stand for drinks. This shows how students were able to use letters and other representations as variables to stand for quantities and unknown amounts. In many of their explanations, students included words like "subtracted from *both sides*" or "divided *each side*" showing that they understood the concept of equality. They also showed evidence of using operations such as subtraction and division to find the value of *x* (Figure 11).

CONCLUDING REMARKS

These classroom projects show how the use of different representational forms can contribute to student learning and promote relational thinking. Students' exposure to multiple representations of algebraic relationships allowed them to translate among pictorial, manipulative, symbolic, and written representations and to develop representational fluency. The algebraic explorations used in the unit were motivating to the third graders and fostered their relational and algebraic thinking.

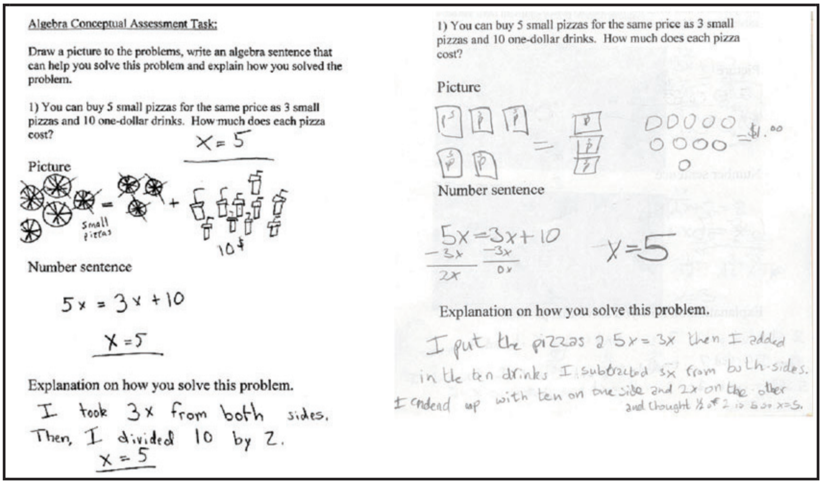


Figure 11. Examples of student solutions on algebra word problems with symbolic and pictorial representations

In addition to the different representational forms promoting student thinking, students used different representational forms to express their thinking. Their drawings, equations, and written explanations show their fluency in translating from one representational form to another, whether they were using the virtual balance scale or the physical balance scale as their primary classroom manipulative for the unit. Moving among different representations helped to strengthen their developing understanding. As the student work samples illustrate, students developed facility in expressing their understanding of numeric relationships through drawings, symbols, and written explanations.

There were unique and distinct features available in both the virtual and physical environments for learning this concept. For example, the activities using virtual algebra applets promoted the understanding of the fundamental algebraic idea of equality using the dynamic feature of the tilting balance scales. The Hands-On Equations® encouraged students' invented methods and mental mathematics.

These results show that different manipulative models, both in the physical and virtual environments, may have unique features that encourage relational thinking and promote algebraic reasoning. This project also illustrates that there are fundamental ideas of algebra, such as equality, the use of variables, and solving for an unknown quantity, that are appropriate for investigation by elementary school students. Different representations, including those increasingly available through technology, can facilitate the teaching of these fundamental ideas.

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